Proceedings of the 5th International Conference on Nonlinear Dynamics
ND-KhPI2016
September 27-30, 2016, Kharkov, Ukraine

Application of R-functions Theory to Nonlinear Vibration Problems of Laminated Shallow Shells with Cutouts

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Abstract
In present work an effective method to research geometrically nonlinear free vibrations of elements of thin-walled constructions that can be modeled as laminated shallow shells with complex planform is applied. The proposed method is based on joint use of R–functions theory, variational methods and Bubnov–Galerkin procedure. It allows reducing an initial nonlinear system of motion equations of a shallow shell to the Cauchy problem. The mathematical formulation of the problem is performed in a framework of the refined first-order theory. The appropriate software is created within POLE–RL program system for polynomial results and using C++ programs for splines. New problems of linear and nonlinear vibrations of laminated shallow shells with cutouts are solved To confirm reliability of the obtained results their comparison with the ones obtained using spline-approximation and known in literature is provided. Effect of boundary condition on cutout is studied.

Keywords
R–function theory, Timoshenko’s theory, laminated shallow shells, geometrically nonlinear vibrations

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Introduction

Investigation of geometrically nonlinear vibrations of multi-layered shallow shells with complex planform and a cutout is carried out. The mathematical statement of this class of problems is well developed in literature [1, 2]. However, due to the complexity of the motion equations system, there are practically no works, which contain numerical results for these problems in the case of complex planforms, the presence of cutouts and different types of boundary conditions. Therefore, development of methods solving such problems is an important task. In this paper we propose a method based on the theory of R-functions [3, 4] and variational methods, which allows one to solve such problems.

1. The mathematical statement of the problem

We consider the multi-layered thin shell of the constant thickness h, on the assumption that the slip and separation between the layers are absent. We confine ourselves to the symmetrical structure of layers. The mathematical formulation of the problem is performed via refined theory of multi-layered shells based on the Timoshenko’s shear assumptions. Then the problem of geometrically nonlinear vibrations of shallow shells is reduced to the solution of the system of nonlinear differential equations of motion [5, 6]:

$$\frac{\partial N_{11}}{\partial x}+\frac{\partial N_{12}}{\partial y}=m_1\frac{\partial^2 u}{\partial t^2};\qquad \frac{\partial N_{22}}{\partial y}+\frac{\partial N_{12}}{\partial x}=m_2\frac{\partial^2 v}{\partial t^2};$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+k_1N_{11}+k_2N_{22}+N_{11}\frac{\partial^2 w}{\partial x^2}+2N_{12}\frac{\partial^2 w}{\partial x\partial y}+N_{22}\frac{\partial^2 w}{\partial y^2}=m_1\frac{\partial^2 w}{\partial t^2}; \qquad (1)$$

$$\frac{\partial M_{11}}{\partial x}+\frac{\partial M_{12}}{\partial y}-Q_x=m_2\frac{\partial^2\psi_x}{\partial t^2};\ \frac{\partial M_{22}}{\partial y}+\frac{\partial M_{12}}{\partial x}-Q_y=m_2\frac{\partial^2\psi_y}{\partial t^2},$$

where $u(x,y,t), v(x,y,t), w(x,y,t)$ are displacements of the coordinate surface points; ψ_x,ψ_y are rotation angles of the normal to the coordinate surface; N_{11}, N_{12}, N_{22} are the in-plane resultants per unit length; M_{11}, M_{12}, M_{22} are the internal moment resultants per unit length; Q_x, Q_y are the transverse shear resultants per unit length. Components of these resultants are defined as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix}=\begin{bmatrix} C_{11} C_{12} C_{16} K_{11} K_{12} K_{16} \\ C_{12} C_{22} C_{26} K_{12} K_{22} K_{26} \\ C_{16} C_{26} C_{66} K_{16} K_{26} K_{66} \\ K_{11} K_{12} K_{16} D_{11} D_{12} D_{16} \\ K_{12} K_{22} K_{26} D_{12} D_{22} D_{26} \\ K_{16} K_{26} K_{66} D_{16} D_{26} D_{66} \end{bmatrix}\cdot\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y}+\psi_{y,x} \end{bmatrix},\quad \begin{aligned} Q_x &= k_5^2 C_{55}\left(\frac{\partial w}{\partial x}+\psi_x\right)+k_4^2 C_{45}\left(\frac{\partial w}{\partial y}+\psi_y\right), \\ Q_y &= k_5^2 C_{45}\left(\frac{\partial w}{\partial x}+\psi_x\right)+k_4^2 C_{44}\left(\frac{\partial w}{\partial y}+\psi_y\right). \end{aligned} \qquad (2)$$

The expressions of deformations, taking nonlinear terms in account, are expressed as

$$\varepsilon_x=\frac{\partial u}{\partial x}-k_1 w+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\quad \varepsilon_y=\frac{\partial v}{\partial y}-k_2 w+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\quad \varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.$$

The expressions (2) contain coefficients C_{ij}, K_{ij}, D_{ij}, m_1, m_2, which are calculated by the known formulas [7-10], k_4, k_5 are shear correction coefficients, k_1, k_2 are curvatures of a shell.
The system (1) is supplemented by appropriate boundary and initial conditions.

1.1 The solution method

Accordingly to the algorithm proposed in [8-10], we represent the unknown functions $u(x,y,t), v(x,y,t), w(x,y,t), \psi_y(x,y,t), \psi_x(x,y,t)$ in the form of an expansion by eigenfunctions of the corresponding linear problem solution of free vibrations of shells

$$w(x,y,t)=y_1(t)\cdot w_1(x,y),\ \psi_x(x,y,t)=y_1(t)\cdot\psi_{x_1}(x,y),\ \psi_y(x,y,t)=y_1(t)\cdot\psi_{y_1}(x,y),$$

$$u(x,y,t)=y_1(t)\cdot u_1(x,y)+y_1^2(t)\cdot u_2(x,y),\ v(x,y,t)=y_1(t)\cdot v_1(x,y)+y_1^2(t)\cdot v_2(x,y). \qquad (3)$$

The functions $u_1, v_1, w_1, \psi_{x_1}, \psi_{y_1}$ are the components of the eigenvector $\bar{U}=(u_1, v_1, w_1, \psi_{x_1}, \psi_{y_1})$, corresponding to the first mode. The functions u_2, v_2 must be a solution of the system of differential equations

$$\begin{cases} L_{11}u_2+L_{12}v_2=-Nl_1^{(2)}(w_1); \\ L_{21}u_2+L_{22}v_2=-Nl_2^{(2)}(w_1). \end{cases} \qquad (4)$$

The expressions for the right-hand sides of (4), designated by operators $Nl_k^{(2)}(w_1), (k=1,2)$ are described in [8-10].

The system of equations (4), supplemented with the appropriate boundary conditions, as well as a linear problem of free oscillations of multi-layered shallow shells, can be solved using RFM method [3, 4] for almost any form of the shell's plan and various types of boundary conditions.

Substituting the expression (3) for the unknown functions in the system (1) and applying the Bubnov-Galerkin procedure, we obtain the nonlinear ordinary differential equation

$$y_1''(t)+\omega_l^2 y_1(t)+\beta\cdot y_1^2(t)+\gamma\cdot y_1^3(t)=0. \qquad (5)$$

Formulas for the coefficients β, γ presented in the equation (5) are given in [8-10].

The solution of the obtained ordinary differential equation may be accomplished through a variety of approximation methods. For example, the Runge-Kutta method, Bubnov-Galerkin and others.

2. Numerical results

To validate the proposed method and created software a number of test problems were solved. The obtained results were compared with the once of other authors [7, 11]. One of the test cases is discussed in Example 1.

Example 1. Consider the problem of free vibrations of a shallow geometrically nonlinear isotropic square shell of constant thickness of double curvature. The following geometric and material parameters are used: $a/b=1; R_x/R_y=1; R_x/R_y=0;$ $R_x=10; h=0.01; \nu=0.3$. The shear correction factors are: $k_4^2=k_5^2=5/6$. The boundary conditions correspond to a simply-supported edge. Figure 1 shows a comparison of the frequencies ratios dependence on the basic mode amplitude for cylindrical and spherical shells, obtained by RFM with the known results of [11]. To obtain the dependency ratio of the frequencies to the amplitude the Runge-Kutta method was used.

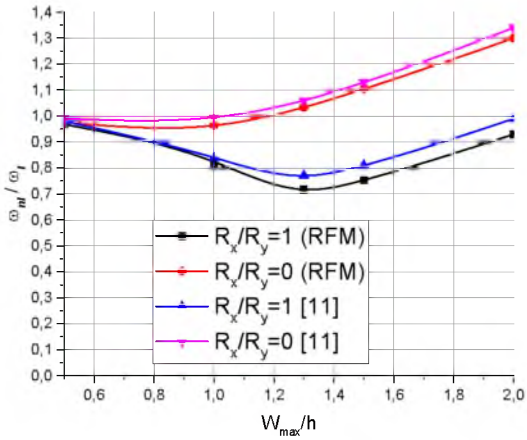

Figure 1. Convergence of amplitude-frequency curves of shallow shells

The observed divergence of obtained results with the once of [11] does not exceed 2%.

Example 2. Consider the problem of a 3-layered cylindrical shallow shell free nonlinear vibration with a square planform and a central square cutout (fig. 2).

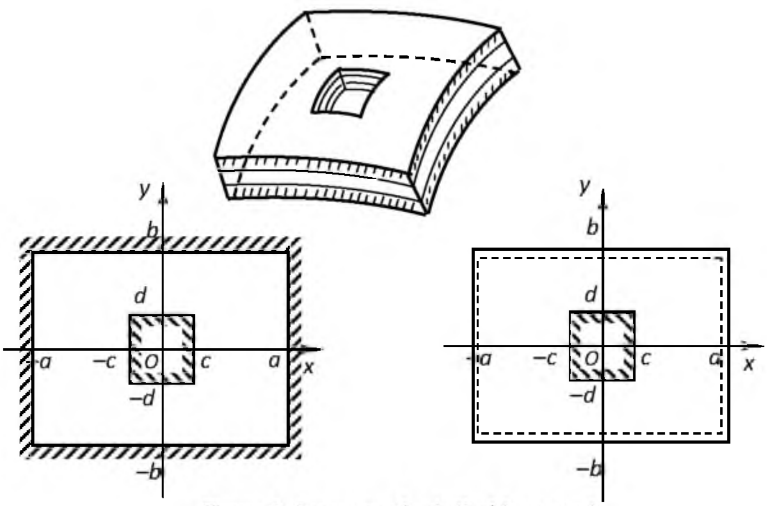

Figure 2. Geometry of a shell with a central cutout and different boundary conditions

It is assumed that all layers are made of the material with the following characteristics:
Material 1 (M1): $E_1 = 40E_2, G_{12} = G_{13} = G_{23} = 0.5E_2, \nu_{12} = 0.25.$
Material 2 (M2): $E_1 = 10E_2, G_{12} = G_{13} = G_{23} = 0.5E_2, \nu_{12} = 0.25.$

The shear correction factors are taken as $k_4^2 = k_5^2 = 5/6$, dimensionless parameters of the curvature are defined as $R_x / h = 300, h/(2a) = 0.01$. Three types of boundary conditions are investigated: completely clamped edge (CC), simply supported on external edge and clamped cutout (SC), simply supported on external edge and free cutout (SF). Comparison of the obtained fundamental frequencies for two types of materials (M1, M2) with the once in [7] is presented in table 1. Results presented by RFM were obtained both by polynomial approximation (POLY) and splines [12] (SPLI) using mesh of 10x10.

Table I Comparison of the non-dimensional frequency $\varpi_1 = \omega_1 a^2 \left(\rho / E_{22} h^2\right)^{1/2}$ for cross ply $(0/90/0)$ laminated shells with the boundary condition (SF).

c/a	M1	M1	M2	M2
0	29.3565 [7]	29.481 [POLY]	23.8253 [7]	23.933 [POLY]
		29.476 [SPLI]		23.921 [SPLI]
0.2	29.4182 [7]	30.252 [POLY]	24.2845 [7]	24.749 [POLY]
		29.859 [SPLI]		24.552 [SPLI]

The observed divergence of obtained results with the once of [7] does not exceed 3%.
Further, new results are presented by using the theory of R–functions.
The amplitude-frequency dependence for cylindrical shells of SC boundary condition with the cutout of size $c/a = 0.2$ for two types of materials (M1, M2) is presented in figure 3.

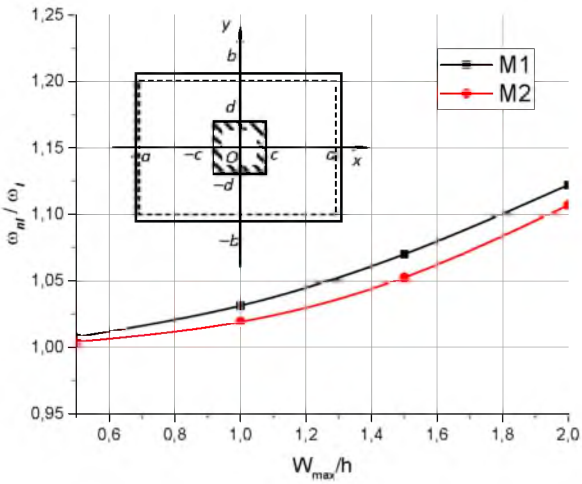

Figure 3. Amplitude-frequency curves of SC cylindrical shallow shells

According to the observed curves we can state that the behavior of investigated shell of M1 material is more rigid than the one of M2 with the amplitude increase.
The amplitude-frequency dependence for cylindrical shells of CC boundary condition with the cutout of size $c/a = 0.2$ for two types of materials (M1, M2) is presented in figure 4.
According to the observed curves we can state that the investigated shell of M1 material becomes more rigid than the one of M2 when the ratio $W_{\max}/h$ exceeds 1.4.

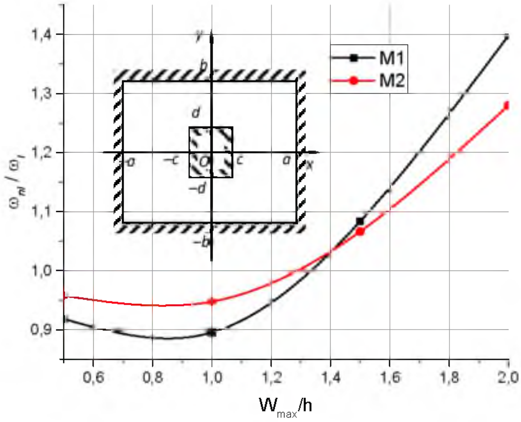

Figure 4. Amplitude-frequency curves of CC cylindrical shallow shells

Conclusions

A proposed numerically-analytical approach based on R-functions theory is used to research free nonlinear vibration problems of laminated shallow shells with cutouts. Three-layered shells made of different materials with different curvatures and square cutout are investigated. Different types of boundary conditions are examined. The amplitude-frequency curves of vibrations of considered shells have been constructed using the first-mode approximation by the Runge-Kutta method. A comparison with known results confirms the reliability of the proposed approach.

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